DOCUMENT RESUME

ED 255 392 SE 045 496

AUTHOR Lampert, Magdalene
TITLE Understanding, Doing, and Teaching Multiplication.
INSTITUTION Michigan State Univ., East Lansing. Inst. for Research on Teaching.
PUB DATE 1 Feb 85
NOTE 82p.
AVAILABLE FROM Institute for Research on Teaching, Michigan State University, Erickson Hall, East Lansing, MI 48854 ($3.50 each).
PUB TYPE Reports - Research/Technical (143) -- Guides - Classroom Use - Guides (For Teachers) (052)

EDRS PRICE MF01/PC04 Plus Postage.
DESCRIPTORS *Algorithms; *Cognitive Processes; Computation; Educational Research; Elementary Education; *Elementary School Mathematics; Interviews; Manipulative Materials; Mathematical Concepts; *Mathematical Models; Mathematics Curriculum; *Mathematics Instruction; *Multiplication; Number Concepts; Problem Solving
IDENTIFIERS *Mathematics Education Research

ABSTRACT

The concept of multiplication is described and illustrated using several different representational systems. A conceptual approach to teaching mathematics is compared with the procedural approach commonly found in the school curriculum. Four different methods of representing the multiplication process with numbers larger than ten are presented: combinations of coins, stories in which quantities are counted by grouping them, recording numbers in a way that is congruent with operations on quantities, and using multiplication as a procedure to solve problems involving estimations of weights. Each of these methods is related to the conventional algorithm for multiplying large numbers. Children's work is used to illustrate how they respond to each approach. (Author)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED255392

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

✓ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

Understanding, Doing, and Teaching Multiplication

Magdalene Lampert
Institute for Research on Teaching
Michigan State University
February 1, 1985

SE 045 496

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
Magdalene Lampert

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

Abstract

The concept of multiplication is described and illustrated using several different representational systems. A conceptual approach to teaching mathematics is compared with the procedural approach commonly found in the school curriculum. Four different methods of representing the multiplication process with numbers larger than ten are presented: combinations of coins, stories in which quantitites are counted by grouping them, recording numbers in a way that is congruent with operations on quantities, and using multiplication as a procedure to solve problems involving estimations of weights. Each of these methods is related to the conventional algorithm for multiplying large numbers. Children's work is used to illustrate how they respond to each approach.

This paper is about multiplication: what it is and how to do it. It is also about how multiplication is usually taught in school and about how it could be taught. My observations about how it could be taught are based on my own current attempts to teach multiplication to a heterogeneous group of 25 fourth graders in a public school classroom.

Examining the teaching and learning of multiplication in school requires attention to how we use numbers and symbols as well as other media to represent mathematical processes. Learning how to manipulate numbers and symbols has traditionally been a substantial part of the mathematics curriculum. My teaching is based on the assumption that it is important for those numbers and symbols to have meaning for children if they are going to go on to study higher mathematics with confidence and if they are going to be able to use mathematical ideas to solve problems in the real world. I also believe that if children understand what numbers and symbols mean, they will become more proficient in the sort of manipulation of numbers and symbols that is highly valued in schools.

Most fourth grade children have probably been introduced to the idea that 7 x 6 means seven groups of six, and they can probably speak this "grouping" language about multiplications up to 10 x 10. When larger numbers come into play, however, it is much harder for children to keep in mind that multiplication is simply a way of counting the total number of objects in situations where those objects are arranged in groups and all of the groups contain the same amount of objects. One can easily picture in one's mind, (and the teacher can easily draw on the board) seven groups of six objects, and everyone can count the total number to get the answer

BEST COPY AVAILABLE

to 7 x 6. When you begin to consider the question asked by 82 x 352, however, it becomes much more difficult to produce an image that will lead to an efficient counting procedure for finding "the answer."

For practical, conceptual, and developmental reasons teachers usually stop using objects to represent mathematical processes in second or third grade. That is when the numbers get bigger, making the objects more unwieldy; it is also when the connection between objects and symbols becomes more complex. Concrete activities with older children become more difficult to manage for social reasons, and at the same time, the children themselves are beginning to be able to understand ideas that are not directly tied to the physical manipulation of objects. But the jump from physically counting out the objects in 82 groups of 352 to computing the total by the steps in the conventional multiplication algorithm is just too much for many children to handle. They see the counting and the computing as separate and unrelated activities, and it is therefore very difficult for them to assign any meaning to 82 x 352. They may learn to <u>compute</u> 82 x 352 successfully using the conventional algorithm, but they probably will not be able to make the connections between the symbols and the meaning of the mathematical processes.

Starting from the activity of counting out 82 groups of 352 objects as the most concrete level of representing 82 x 352, we might think of drawing a picture of 82 groups of 352 objects as a somewhat more abstract, and telling a story in which those numbers and the process of counting them are associated with mental images as even more abstract. Consider, for example, the mental imagery which results from the following story: "Eighty-two new apartment houses have just been built on the north side of the city and each building has exactly 352 apartments in it." Such a

story enables children who are familiar with apartment houses to get at an idea of what 82 x 352 might mean in terms of quantity without counting out 82 groups of 352 or seeing a picture of 82 groups of 352.

A still higher level of abstraction would be writing the number 352 down 82 times and adding to find the total. Further abstracting, we could say that 3520 is 10 groups of 352, and if I added eight 3520's I'd have 80 x 3520 because there are eight 10's in 80. And then if I add 352 two more times to that, I'd have the total. This procedure takes apart and puts together numbers rather than counting objects, but it is still easy to "picture" the procedure that is used to find the total:

82 groups of 352

- 80 groups of 352
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520
  - 10 groups of 352 = 3520

  28160
- 2 groups of 332

  352 + 352 = 704

total: 28160 + 704 = 28864

At the most abstract level, we have the conventional algorithm:

$$\begin{array}{r} {\scriptstyle 4\ 1} \\ 352 \\ \times\ 82 \\ \hline 704 \\ 2816 \\ \hline 28864 \end{array}$$

I have called this level "most abstract" because it is very difficult to connect the steps in this computational procedure with the idea of counting quantities in groups. What are those little 1's and the little 4 on top of the 352? Why is that 2816 moved further to the left than 704? Does the 6 in 2816 stand for 6 or 60? Why do I start by multiplying 2 x 2 and put the answer down under the 2? These questions would probably be

easier to answer if one had a firm understanding of the idea that 82 x 352 means 82 groups of 352, although answering them would require some fairly sophisticated knowledge about how we symbolize "place value."

Very few mathematics teachers or textbooks go through the process of representing multiplication to children at the various levels of abstraction I have described. Most often, they proceed directly from concretely representing single digit multiplication like 7 x 6 as seven groups of six objects to teaching children the procedures for multiplying 82 x 352 using the conventional algorithm. Sometimes the grouping process is pictured in textbooks, but very little is said about "groups" once children are multiplying with two numbers that are both bigger than 10.

After they have learned to use the conventional procedures to do these multiplications, students might be asked to solve word problems like: "If baseball shirts cost $8.95 each, what would be the total cost of 73 shirts?" But doing such problems does not necessarily indicate an understanding of the algorithm. Children can learn to recognize problems like this as members of a set of "word problems" to which multiplication should be applied (usually because they come at the end of the chapter on multiplication), but they probably could not tell you why multiplying gets you the correct answer. If they understood that such a problem was asking them to find the amount of money in 73 groups of $8.95, they would probably also be able to answer more practical questions like: "If there were 900 dollars in the school budget for 73 baseball shirts, would we have enough money to buy them?" Understanding multiplication as a process of counting by grouping has at least as much practical value as being able to compute the exact answer to 73 x 895 precisely using the conventional algorithm.

In addition to its practical value, such understanding can give children the confidence to make the transition from figuring out why arithmetic makes sense to figuring out how to do higher mathematics. When mathematics begins to be nothing more than a set of rules for manipulating symbols that "magically" turn up the right answer, children stop seeing themselves as capable of figuring out why one procedure for moving numbers around makes any more sense than another. Once this happens the only successful learning strategy available to them is to <u>remember</u> what the teacher or the book said you were supposed to do. What gets into their memory then, is a list of rules that are not tied together in any way that would help them to use the right rule on the right occasion. The more mathematics one is taught, the longer the list of rules to remember. It would not be surprising if there were a point where it just gets to be too much. For some children, the algorithm for the multiplication of large numbers is the beginning of the end; others can remember everything up to the process for doing long division. Some make it through remembering the rules for the addition and subtraction of fractions, but the "invert and multiply" rule for dividing fractions is the last straw. "Turning numbers upside down???" "How could that possibly make sense?"

Usually by the time they get through high school, many people (including some who go on to become teachers) are confirmed in the belief that mathematical rules generally <u>do not</u> make sense and that mathematics is an esoteric body of memorized knowledge enjoyed by a small but strange minority. There is hardly time in most teacher education programs to seriously challenge these beliefs, and so many children are taught mathematics by teachers who see the subject as something quite different

than what it really is. The mathematics textbook is treated by these teachers like a cryptic document: teacher and student together engage in puzzling out "what IT wants you to do."

At the point of beginning to learn multiplication, many fourth graders still have the attitude that what they are learning is supposed to make sense to them. How can we teach them in a way that supports this attitude? In any particular group of fourth grade children there will be many levels of ability to understand the multiplication process and a great variation in whether objects in groups, a picture, a story, or operations on numbers with symbols make the most sense as a representation of that process. One way to address this variety in one's audience is to help them see all of the connections: between objects and stories, between pictures and stories, between pictures and numbers, and so on. Developmental studies suggest, in fact, that most fourth graders are in a transitional phase; sometimes they will be able to draw meaning out of abstract numbers and symbols, but often they will need some help "telling a story" when they are presented with a set of numbers and symbols like 46 x 19. Sometimes they will even need to picture actual objects in order to figure out what is happening. The bigger the numbers, the harder it is for children to make connections among different levels of representation. This difficulty arises partly because the exercise of representing arithmetic operations in a meaningful way is not familiar to them, partly because their stories may need to be grounded in mental pictures and it is harder to picture big numbers, and partly because their own lives involve more experience with quantities like 4 groups of 2 or 7 groups of 3 than with quantities like 46 groups of 19.

If one assumes that learning is the active building of schemas for "making sense" by the learner, then it would be important for children to have the experience of making up some of their own stories and pictures about multiplication and testing those stories and pictures against their own already existing mathematical knowledge*. In addition to giving them the opportunity to make sense of mathematical ideas in their own terms, having children make up their own stories has the benefit of turning a rather bland "question" like 86 x 94 = ? into an interesting "problem" that might be fun for them to solve. Stories (like the one about the apartment houses above) evoke mental pictures and may even be accompanied by captivating drawings.

In my fourth grade class, I have been doing several different sorts of lessons which are intended to help children make sense of the process of multiplication with numbers larger than 10, including telling and illustrating stories, working on problems whose solutions require different ways of grouping and counting, and solving problems drawn from "real world" situations in which multiplication is a useful tool. The following sections describe these approaches to understanding, doing, and teaching multiplication.

---

*I'm not sure how T.V. can foster this active schema building except by modeling it, but I believe modeling it could be very powerful since it can make "sense making" part of the kid culture for learning mathematics, e.g. portraying dialogue like "I think it means . . ." "No I think it means . . ." and then an argument about what makes sense. This kind of a culture will not be easy to establish, given what goes on in school, but I have reason to believe it's possible based on my attempts to do it in the classroom where there are strong sanctions against individuals trying to make sense of things themselves rather than checking with a book or a teacher to see whether the "answer" is "right."

Multiplication Stories

Each of my lessons on multiplication stories began with a multiplication written on the board in the conventional manner, e.g. 6 x 7 or $\begin{array}{r} 83 \\ \times\ 2 \end{array}$ or $\begin{array}{r} 936 \\ \times\ 13 \end{array}$. Sometimes the stories generated by these numbers and symbols were invented by me, but most often they were invented by the children. We started with stories that would be representations of multiplying a one digit number by a one digit number e.g., "There were six parties last week and seven children attended each party." as a story for 6 x 7. These came easily; the children I was teaching had done them in third grade. We quickly progressed to stories that would represent the multiplication of a two digit number by a one digit number, and then to stories for two digits times two digits. All of these multiplications could be represented by the same sorts of stories as the one digit by one digit multiplications, but I also invented stories which would represent the process of taking apart the bigger numbers to make them easier to multiply and count in groups. (This process of "taking apart" will be explained more fully below.) During the lessons, I drew pictures on the board to go with my own stories or with a child's story, and on a few occasions, the children made up their own stories individually, wrote them down, and drew pictures to represent them.

What follows is a sample dialogue from one of these lessons. (T means teacher.)

On the board:

12 x 4

T: Can anyone give me a story that could go with this multiplication?

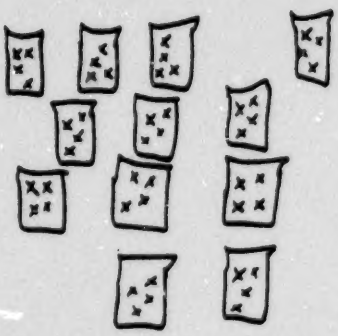
Jessica: There were 12 jars and each had four butterflies in it.

T: And if I did this multiplication and found the answer, what would

I know about those jars and butterflies?

Jessica: You'd know you had that many butterflies all together.

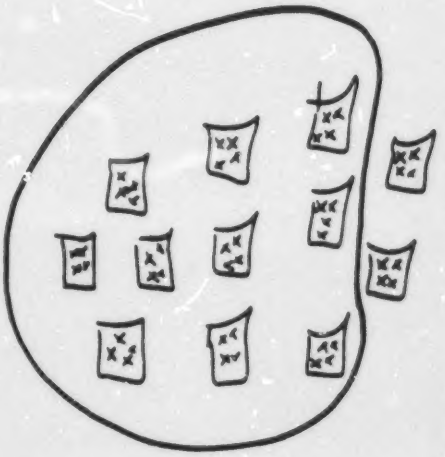
(Note: Jessica has constructed a way of giving meaning to 12 x 4. The meaning, i.e. "what I would know if I did it" is not the same as the answer, i.e. "48." From this point on in the lesson, I am going to be structuring a representation of Jessica's story that focuses on the idea of counting by taking groups apart and putting them together. I will "check in" with different members of the class as I work at the board to keep them attentive and see if they are getting the meaning and the structure I'm trying to convey.)



T: Okay, here are the jars. The x's in them will stand for the butterflies. Now, it will be easier for us to count how many butterflies there are altogether, if we think of the jars in groups. And as usual, the mathematician's favorite

BEST COPY AVAILABLE

number for thinking about groups is? (Draw a loop around 10 jars.)



Sally: 10

T: Each of these 10 jars has 4 butterflies in it, so how many butterflies are inside this circle?

John: 40

T: How'd you figure that out?

(Note: I almost <u>always</u> follow an unexplained "answer" with that question when I am teaching, partly to see what the child will say, partly to give them the opportunity to verbalize their reasoning, and partly to establish the culture of "sense making" as part of our relationship in the classroom.)

$4 \times 10 = 40$

John: It's 4 x 10. That's easy, you just add a zero.

(Note: This is a recognition of a pattern observed from earlier experiences with multiplying numbers by 10. It is a very useful "fact" to have in mind when you are multiplying large numbers. See page .)

T: I put the jars in groups of 10 because I knew it would be easy for you. How many more butterflies are there outside the circle?

Jim: Eight. (He can count them easily now that there are only a few.)

$4 \times 2 = 8$

T: I add ten jars and two jars and I get 12 jars. Each jar has 4 butterflies in it. (Point to the two fours in 4 x 10 and 4 x 2.) So how many butterflies are there altogether?

Chorus: 48

T: I added the 10 and the 2 to get 12 jars. Should I also add the 4 and the 4 to get 8 butterflies?

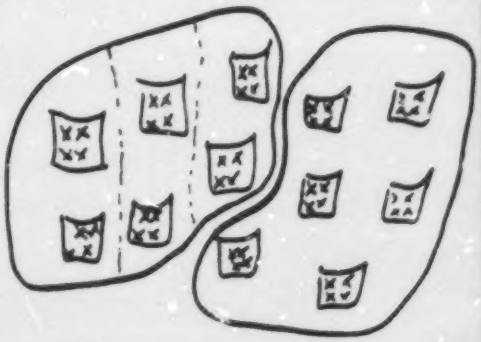
$4 \times 12 = 48$

Shawn: No. There are just four butterflies in each jar. That will never change.

(Note: This is the glitch in what mathematicians call the "distributive law"; it's an easy principle to see when it is attached to quantities in stories like this one, but very confusing when presented in the abstract. When students see only (4 x 10) + (4 x 2) it is very hard to explain why the answer is <u>not</u> obtained by doing 8 x 20. Yet math books often ask children to do such figuring without referring to any representation of quantities.)

T: Suppose I erase my circle and go back to looking at the 12 jars again all together. Is there any other way I could group them to make it easier for us to count all the butterflies?

(Note: This part of the lesson is intended to get at the idea that what is important is finding a grouping that makes the figuring easier. Usually that means grouping by "tens" and "ones," but not always.)



Jean: You could do six and six.

T: Now how many do I have in this group?

Steve: 24

T: How'd you figure it out?

Steve: 8 and 8 and 8. (He put the 6 jars together into 3 pairs, intuitively finding a grouping that made the figuring easier for him.)

T: That's three times 8. It's also 6 x 4.

T: Now how many are in this group?

Jean: 24. It's the same. They both have 6 jars.

T: And now how many are there altogether?

Patty: 24 and 24 is 48.

(Note: Jean's and Patty's answers are what I would call "explained." I got a sense of what made sense to them here without having to ask.)

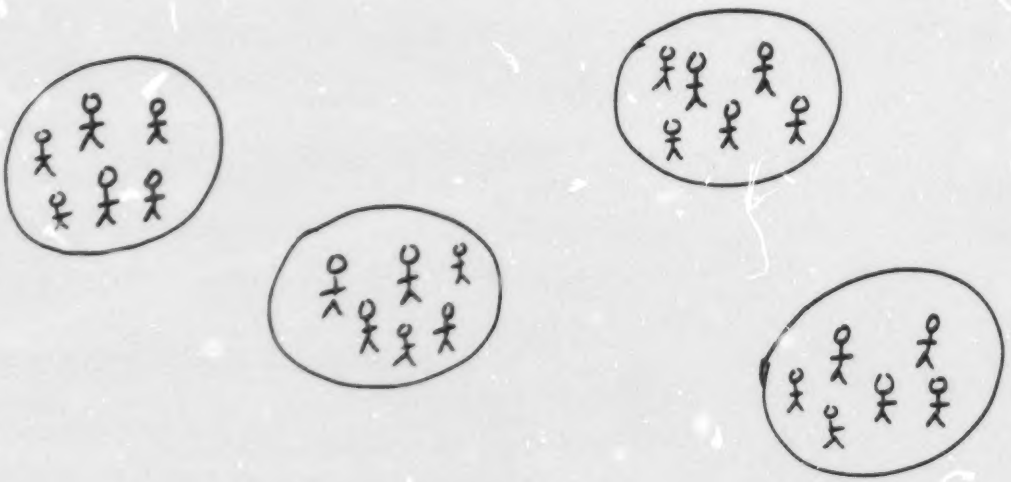
T: Do we get the same number of butterflies as before? Why?

Patty: Yeah, because we have the same number of jars and they still have 4 butterflies in each.

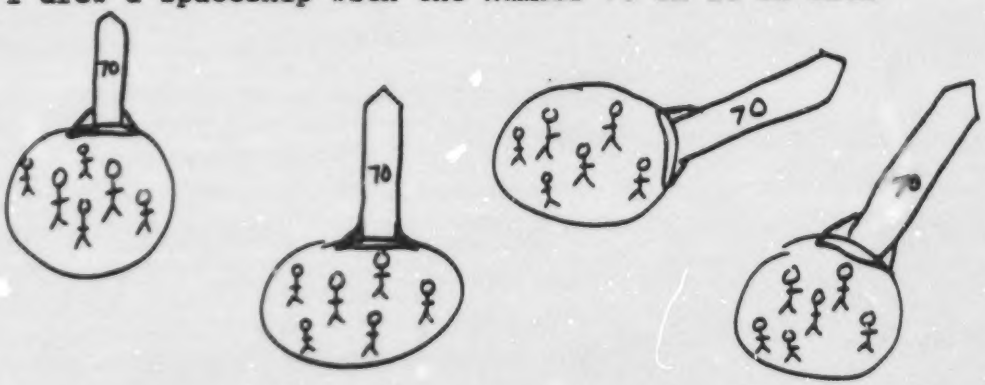
(Note: I asked several children to to explain in their own words why there were the same total number of butterflies each time. It was clear from watching them that some of the kids were surprised that it came out the same, which was cue to me to do lots more of these different kinds

of groupings. The underlying assumption for the conventional algorithm is that 14 x 2 is the same as 4 x 2 plus 10 x 2, yet when they are learning it, kids get very little chance to check and see if they <u>are</u> the same.)

When I made the numbers in the multiplication bigger, I then became more didactic and said I was going to tell them a story to help them think about 76 x 4. I chose a theme from some stories they had told in another context and I told them, "There were four planets and each had six astronauts exploring on it." I drew the following on the board:



Then I said, "Then a big spaceship with 70 more crew members landed on each planet. I drew a spaceship with the number 70 on it on each "planet."



$4 \times 76 = (4 \times 6) + (4 \times 70)$

Space explorers on the planets already

more Space explorers arriving in Space ships

I asked how many space men were <u>on each</u> planet now, and then I asked them to figure out how many space men there were <u>altogether</u>, on all four planets. Some children added 76 + 76 + 76 + 76. Another doubled 70, doubled 140 and got 280 and then added 24 to it to get a total of 304. Another multiplied 70 x 4 and then multiplied 6 x 4 and added the products together. In each case, I had the child explain to the whole class "how they figured it out" and gave equal praise to each solution.

The students were always very eager to make up stories. Some tried hard to make up "real" stories, i.e. stories that might match something that would actually happen, and others delighted in making up stories that were silly but mathematically appropriate. After we did a few more variations on the theme of grouping and totaling over several days working as a class, I decided to have them try individually to make up stories and draw pictures for multiplications of two-digit numbers by one-digit numbers. In order to structure their work productively and help them tie their thinking into the conventional structure of "decomposition" into tens and ones, I divided the assignment into four parts: directions, stories, pictures, and calculations. I gave the directions in the following form: "Find 3 x 4 and 50 x 4 and then add them together to get 53 x 4." In other words, I was doing the decomposition of 53 groups into 50 groups and 3 groups for them, and doing it along "place value" lines. They were supposed to invent stories and pictures to go along with these

directions and do the calculations to find the total. They were not using the stories and pictures to help them invent a strategy to find the total in this exercise; they were simply representing the arithmetical processes in a more concrete media.

Before they began to work individually, we did one problem together as a class with me writing and drawing on the board to provide a model for their individual work. One of the boys invented the following story/question to go with the steps to finding 53 x 4: "There were three planets that each exploded into four parts. In another part of the universe, there were 50 more planets that each exploded into four parts. How many new planets are there?" I drew enough on the board for them to get the idea of what was going on here (not 50 planets!). I then set them to work individually to write a story, draw a picture and do the calculations for another set of directions: "Find 20 x 3 and then find 8 x 3, and then add them together to find 28 x 3." They worked on this in class, where I could observe what they were doing and converse with them about it. Then they were assigned a similar task for homework, given the directions: "Add 4 x 3 and 10 x 3 to find 14 x 3." These two assignments gave me some sense of the level of the children's understanding of the place-value decomposition process that underlies the procedures we usually use to do multiplication and they gave them the opportunity to represent these processes in their own terms.

Most of the children focused on the calculations first: if they were figuring out 28 x 3 they would do 20 x 3 = 60 and 8 x 3 = 24 and then add 60 and 24. Some of their stories and pictures represented only on the last part of these calculations and not the groupings, i.e. they told "adding stories" rather than "multiplication stories." (See attached

figures 1 and 2 for examples of this approach.) One girl further "decomposed" 20 x 3 into 10 x 3 and 10 x 3 to make the numerical calculations easier, but her story and picture do not represent this decomposition. (fig. 3) Some children were able to follow the directions well enough to do the correct calculations but their stories and pictures did not match the mathematical processes symbolized by the way they manipulated the numbers. (See attached figure 4, for example.) There were several stories which did represent the calculations completely and appropriately, however, in the manner of figure 5 and figure 6. One girl did a variation in which she told a story that represented her own decision to add 10 plus 4 to get 14 and then multiply 14 x 3 to get 42. (See figure 7.)

After they did these individual productions, I did several more "whole class" lessons in which one student was called upon to invent a story for the numbers and symbols which I wrote on the board. As they told stories, I paid particular attention to the children's use of words, helping them to formulate stories which represented the idea of two groups being joined together into a third group which encompassed everything in both. For example, one girl told the story, "There were four monkey bars out on the playground and six kids were playing on each. There were also 50 swing sets, with six kids on each. How many all together?" I questioned and prodded her until she changed her question at the end to, "How many children were playing on the playground equipment (i.e. a larger category that encompasses both monkey bars and swings) all together?"

Throughout these story-telling activities I emphasized a step-by-step approach to representing the numerical directions, doing the two multiplications first and then the addition, which is the order in which

BEST COPY AVAILABLE

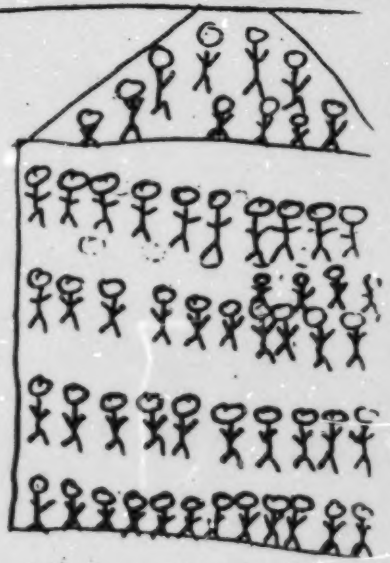
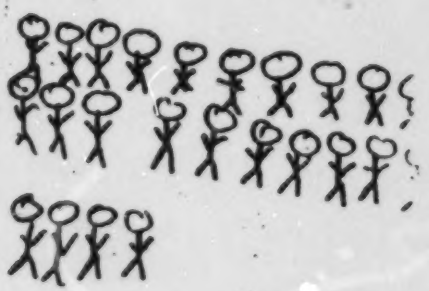
Directions	calculations	Story	Picture Figure 1
Add 10x3 and 8x3.	20x3=60 8x3 = 24 > 84	60 people were at a party 24 new people came to the party. How many people were at the party?	

Figure 1



BEST COPY AVAILABLE

Figure 2

Directions

1. 20x3
and
8x3

Calculations

20 x 3 = 60
8 x 3 = 24

then you add
60 to 24

$$\begin{array}{r} 60 \\ +\ 24 \\ \hline 84 \end{array}$$

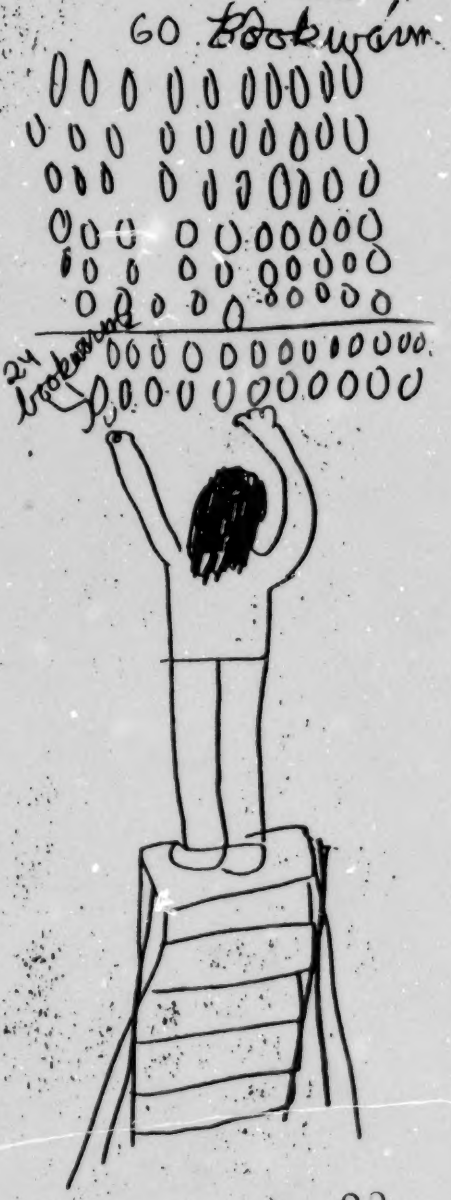
Name of data
Date 1-29-89

Figure 4

Story

There were
60 bookworms
on the wall
and then 24
more got put
up. Now
many were
on the wall

Picture



BEST COPY AVAILABLE

Directions

~~Add~~

~~3 0 3~~

~~2 4 2~~

~~1 2 3~~

Add
20 x 3
and
8 x 3.

Figure 5

Idrianne
Xaeger

Calculation

3 x 10 = 30
3 x 10 = 30

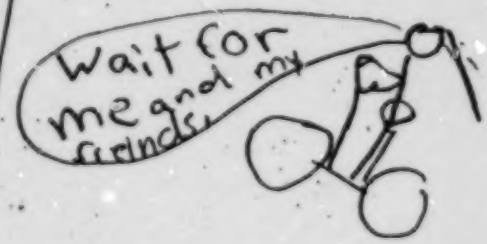
3 x 8 = 24

+30
30
+24
84

Story

60 people were riding bikes, Then 24 more came how many people all together,

Picture Figure 5



BEST COPY AVAILABLE

Add 20 X 3 and 8 X 3

Figure 4

ABA Yank an
Nov 29, 1964.

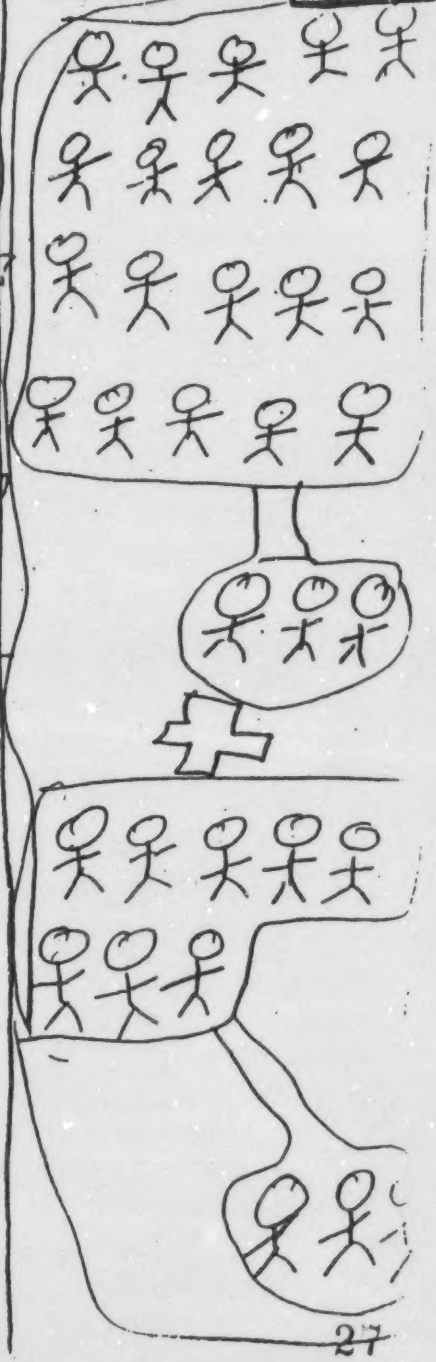
Calculations

20	8	60
x 3	x3	+24
60	24	84

story

There were 20 people talking three more came. how many are there? then there were 8 other groups and 3 more joined the group how many are there all together?

Picture — Figure 9



BEST COPY AVAILABLE

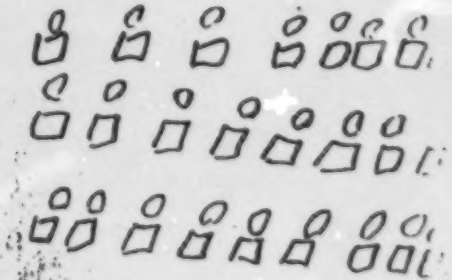
Directions	Story	Calculations	Picture
add 4x3 to 10x3	4 kids in each row and there are 3 rows. In the next class room there are 10 kids in each row and there are 3 rows. How many kids in all?	4 x 3 = 12 10 x 3 = 30 12 +30 42	 room 1  room 2

Figure 5

BEST COPY AVAILABLE

Directions

Add
4X3
to
10×3
Christian
Lipsett

Figure 6

Story

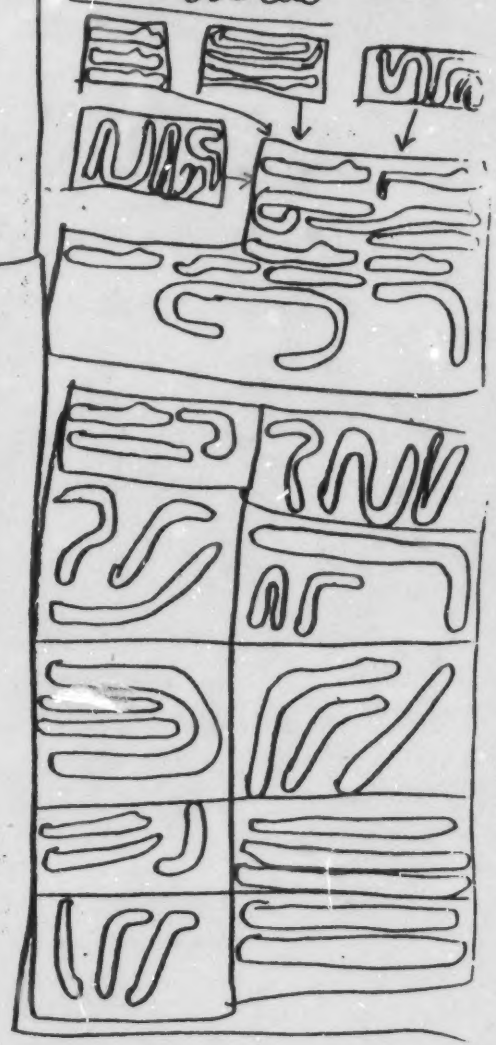
Four houses with 3 boas in each and the keeper put them altogether in one house. Oneother keeper had 10 houses with 3 pythons in each. He too put them in one house. The 2 keepers put them all together in one house. Then the added them all together.

Calculations

4X3=12
10X3=30
30+12=42

Pictures

Figure 6



BEST COPY AVAILABLE

Figure 7

Directions

Add 4x3
too 10x3

Calculations

44
+10
14

14
x3
42

Story

There were 10 planets 4 more came to that solar system. The planets each had 3 moons.
How many moons in all?

Picture

Figure 1

BEST COPY AVAILABLE

they would be done using the conventional algorithm. My purpose was to connect representations that made sense to the children to the symbols used to represent these processes in traditional school mathematics. The brighter kids in the class never got tired of the "story telling" activity, even though they have the mechanical skills to do multiplications like 36 x 42. Everyone got a lot of needed practice in quick recall of their "times tables" as we worked each problem together.

BEST COPY AVAILABLE

Using Numbers in a Meaningful Way to Illustrate the Process of Multiplication

On some occasions, I did not work with stories, but represented the process of grouping and counting in multiplication only with numbers and symbols. The total in 3 groups of 86, for example, could be figured out in four steps and represented as follows:

86	→	80 + 6	Step 1
× 3			
18	←	3 × 6	Step 2
+240	←	3 × 80	Step 3
258	←	18 + 240	Step 4

We did several multiplications in this way together as a class, with me writing on the board and the children doing more and more of the work on their own papers.

I pointed out in every example that there was always a zero on the end of "the answer" in what appears above as step three. When I asked them to speculate about <u>why</u> that always happened, five or six different versions of the "zeroes rule" were produced: they all observed that <u>before</u> you multiply, one of the terms always has a zero at the end, so no matter what you multiply it by, the answer will always have a zero at the end. In

more sophisticated mathematical terms, we would say that since one of the factors is a multiple of 10, the product will also be a multiple of 10. My students have not yet learned the language of factors and multiples, but they have had a lot of experience with multiplying by 10 and multiples of 10, 100 and multiples of 100, 1000 and multiples of 1000.

I did some illustrations of this procedure in which step 3 and step 2 were interchanged, for example:

97	→ 90 + 7	Step 1
× 4		
360	← 4 × 90	Step 2 (or 3)
+ 28	← 4 × 7	Step 3 (or 2)
388	← 360 + 28	Step 4

That these two steps could be done in either order did not seem to cause any problems, and several children were able to say why it made sense to them to be able to switch them around. In fact, the second version made <u>more</u> sense to them when they were able to put the convention of "starting with the ones" aside. When you "take apart" a two digit number in order to read it, you think first of the tens and then of the ones. (I.e. 73 is <u>seventy</u>-three not three and seventy.) It was important for them to be able to explain at this point that some parts of the procedure could be "switched around" and it wouldn't make any difference to the answer, rather than just believing me when I said it. I sometimes referred back

to a story or some drawings so that they had a picture of what the "switching around" looked like and to give them a referent for explaining why it made sense.

This representation for multiplying a two digit number by a one digit number extended easily to representing a one digit times three digit multiplication and beyond: to one digit times four, five, and six digits. So, for example, 3652 x 8 would be:

$$
\begin{array}{rl}
3652 & \longrightarrow 3000 + 600 + 50 + 2 \\
\times\ 8 & \\
\hline
24000 & \leftarrow 8 \times 3000 \\
4800 & \leftarrow 8 \times 600 \\
400 & \leftarrow 8 \times 50 \\
+\quad 16 & \leftarrow 8 \times 2 \\
\hline
29,216 & \leftarrow 24000 + 4800 + 400 + 16
\end{array}
$$

After we did several of these together as a class and they did several individually with success, I showed them the "short cut" notation in which the "answer" appears all on one line; i.e. instead of writing:

$$
\begin{array}{rl}
38 & \\
\times\ 3 & \\
\hline
24 & \leftarrow 3 \times 8 \\
+90 & \leftarrow 3 \times 30 \\
\hline
114 &
\end{array}
$$

we usually write:

$$\begin{array}{r} {\scriptstyle 2}\,3\,8 \\ \times\ 3 \\ \hline 1\,1\,4 \end{array}$$

In the short cut, 90 does not get written down anywhere and 20 is symbolized by the little 2 up on top of the 3 (which is really 30, anyway!)

When the children tried to use this notation, the most common mistake that occurred was the following:

$$\begin{array}{r} 3\,8 \\ \times\ 3 \\ \hline 9024 \end{array}$$

What has happened here is that 3 x 30 became 9000 because there was not enough "room" in the tens place of the answer for both the 9 of 90 (3 x 30) and the 2 of 24 (3 x 8). In order to write all the "answers" on one line, one thing you can do is move over to the left, i.e., to put the 90 to the left of the 24. In order to <u>read</u> these two answers, however, we apply the framework of place value, assigning the name

"thousands" to the fourth place over from the right. What gets written down as 90 thus becomes 9000 when it is read.

I could discern from my conversations with the children who made this "mistake" that they understood and were able to use the basic idea of decomposition as a way to multiply, but they wrote the results of their calculations in a way that did not have the meaning they intended. They did not take account of the fact that in the hierarchy of mathematical symbols, the way we represent place value by lining digits up from right to left is most fundamental. Hence, it <u>does</u> make a significant difference whether you write your products one on top of the other like this: 90
24
or next to the other like this: 9024. The first configuration suggests the addition of 90 and 24, the second does not. It is important to note that the children who wrote down 9024 did not have the "misconception" that 3 x 30 = 9000. They simply did not know how to symbolize <u>the sum of</u> 90 and 24.

One of the questions that often gets raised at this point in math education circles is whether this particular error in symbolization would have developed if I had not taught them to do these kinds of problems by putting the two products down separately rather than by "carrying" from the beginning. Of course, I don't know the answer to that, but I do know that I've seen children invent all sorts of similarly sensible but erroneous symbolizations in classrooms where they were <u>only</u> taught the steps in the conventional algorithm and directed to practice them over and over without any sense of why they work. Children who have a well-developed sense that numbers stand for quantities (and that multiplication is a matter of counting groups) know that counting three groups of 38 will not get you anywhere near 9024. But this sort of number

sense takes a long time to develop and its development is not helped by most of what is taught in elementary mathematics classrooms.

The child who sees 9024 on his or her paper after completing the two multiplications 3 x 8 and 3 x 30 faces a conflict in his or her own understanding. It does make some sense, and yet three groups of 8 and 3 groups of 30 would not add up to 9024. It makes sense if you see 9024 as a row of answers to 3 x 30 and 3 x 8 with no spaces in between. But it does not make sense if you read 9024 as nine thousand twenty-four. How is such a conflict to be resolved? We could tell such a child how to resolve it, but the telling should respect the fact that both ways of looking at 9024 do make some sense.

If the teacher acknowledges that the answer 9024 does make some sort of sense, then the child who came up with it can see him or her self as a sense-maker rather than as a mistake-maker. But one cannot stop there and neglect to point out that this number can be interpreted in two different and conflicting ways. Because of the kind of subject mathematics is, the conflict between the two interpretations must be resolved. In the process, the child can be challenged to build a new "sense-making" schema that will turn up an answer without ambiguity. (Non-Euclidean geometrics as well as other powerful mathematical ideas got invented this way!) Recognizing the conflict may even give learners a better appreciation for why we have adopted the practice of writing the little "2" (from 24) up on top of the 3 in the conventional algorithm instead of writing it underneath the line.

Acknowledging the child's conflict in understanding rather than seeing this kind of situation as a right-wrong answer dichotomy is a way of looking at mathematical knowledge that is very unusual in classrooms and

textbooks. It goes strongly against the grain of the way knowledge is usually organized for teaching and learning in schools. But it also has incredible power for giving the child a different picture of him or her self in relation to the process of learning mathematics. If children can learn to say "that does not make sense to me" and to tell their teachers what <u>does</u> make sense, and if teachers can respond by acknowledging that there is some sense to what the child says, it seems much less likely that so many children will "drop out" of the study of mathematics because they perceive it as something they could never really understand. It is true that some children can succeed for a while in the subject just by learning to <u>do</u> computation even if they don't understand what they are doing or why, but they probably won't get very far beyond high school algebra. They probably also will not see mathematical knowledge as useful or usable.

The place value complexities that arise in multiplying a number larger than 10 by a one digit number become even more problematic for learners when they are taught to multiply two numbers that are <u>both</u> larger than 10. In order to examine the approach I took to teaching children how to connect the meaning of a multiplication like 63 x 24 with the steps one might use to find the product of these two numbers, it is useful to look at such a multiplication in terms of decomposing and recomposing groups.

63 x 24 can be interpreted as 63 groups of 24. Those 63 groups can be separated into two sets of groups: 60 groups of 24 and 3 groups of 24. If we could figure out the total quantity in both of these sets of groups and add them together, we could know how many were in 63 groups of 24. But it's not very easy to see how many are in 60 groups of 24 or how many are in 3 groups of 24, so we need to treat each of these as a separate

"problem," using the same sort of taking apart and putting together as I described above to find how many are in each of the two sets of groups. We can start with either one. How might we break down 60 groups of 24 to better be able to count what is there? We could think of it as 10 large groups each containing 6 groups of 24,and work from there, first finding 6 groups of 24 (2 groups of 24 + 2 groups of 24 + 2 groups of 24) and then multiplying that by 10. Or we could think of 60 groups of 24 as 60 groups of 20 and 60 groups of 4. Because of the "zeroes rule" it is easy to figure out how many there are here: 60 groups of 20 is 1200, and 60 groups of 4 is 240. The total is 1440. Now we need to figure out how many are in 3 groups of 24. One of my fourth graders who had played around a lot with groups and numbers might say, "Well that's the same as 6 groups of 12, and that's 72." You could also think of it as 3 groups of 20 (60) plus 3 groups of 4 (12) for a total of 72.

All of this becomes more clear if the different kinds of taking apart and putting together can be shown in a picture.The most convenient way for books and teachers to illustrate this sort of decomposition is with squared off areas, using the rows and columns to help with the counting.

Total = group A + group B

63 rows of 20 columns + 63 rows of 4 columns

group A = group C + group D

60 rows of 20 columns + 3 rows of 20 columns

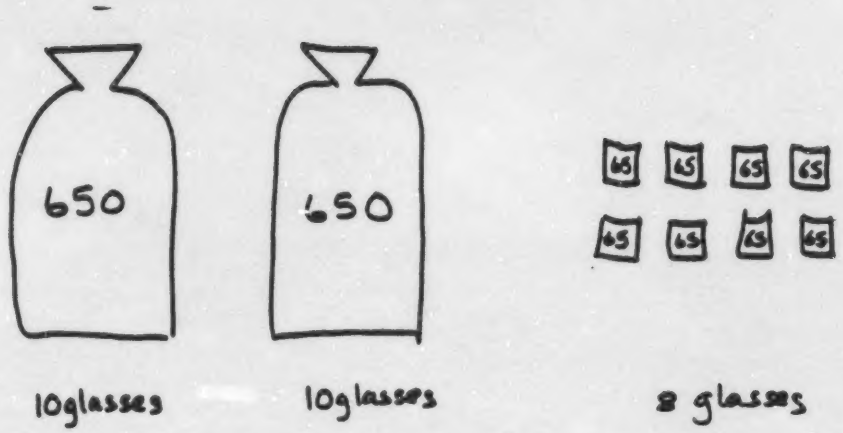
group B = group E + group F

60 rows of 4 columns + 3 rows of 4 columns

But this is still quite an abstract representation for kids, unless their parents work in the tile business. It gets all mixed up with ideas about length of sides and area, concepts introduced around the fourth grade but never taught very well. If you talk about illustrating 2 x 3 for example by , you can say this figure is 2 blocks across and three blocks down, 2 columns of 3 blocks, or 3 rows of 2 blocks, and 6 blocks all together. But many books and teachers do not talk about such an illustration in terms of rows and columns of blocks. Instead, they switch into measures of length and area, saying that such a figure has a width of 2 inches and

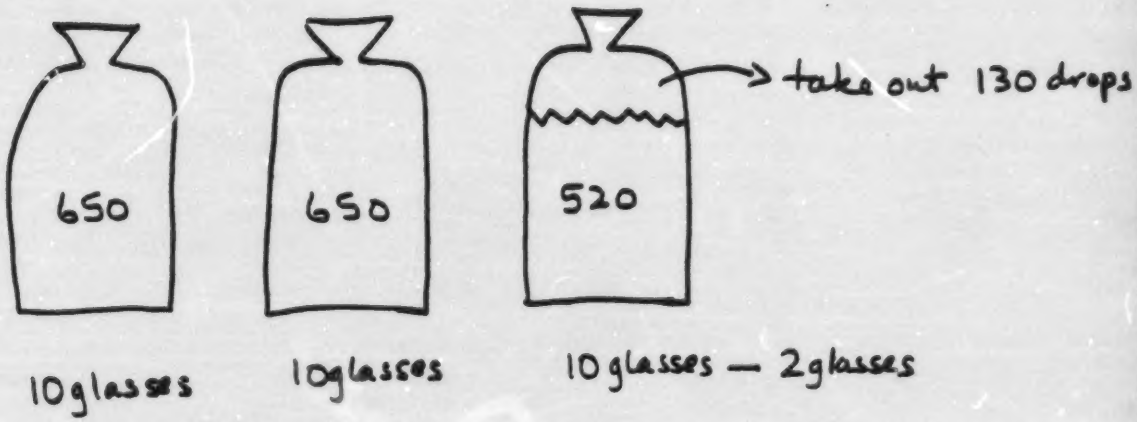
a height of 3 inches and its area is 6 square inches. Now the difference between inches and square inches is not an insignificant difference; one measures length while the other measures area. To try to connect these measures to the ideas about grouping associated with multiplication is difficult: are we talking about 2 groups of 3 inches? or 3 groups of 2 inches? and if so, in either case,how do we wind up with a total of 6 square inches instead of just 6 inches? These complications made me reluctant to use blocks built up into rectangles as an illustration for large number multiplications. One would need to be very careful with the language and checking children's sense of what it all means in order for this to make sense to them as an illustration of the multiplication process.

When I asked my fourth graders to come up with a story about such a problem, we worked out a rather nice pictoral representation together. Given the multiplication 28 x 65, Coleen suggested that we could think of it as 28 glasses with 65 drops of water in each glass. I said I didn't want to draw 28 glasses on the blackboard, so instead I would draw two big jugs that would each hold the same number of drops as 10 glasses. I questioned the students as I drew: How many jugs would I need? Two. And how many more glasses? Eight. How many drops in each glass? Sixty-five. And how much in each jug? Six hundred and fifty. For each of these answers, I requested an explanation, and the student verbalized how he or she "figured it out."



Instead of drawing the drops in each glass, I put numbers in each glass and on the jugs. I asked the class what we should do to find the total number of drops? Add the jugs together. That's 1300 drops. Then I said I would show them a trick that would make it easy to add the drops in all the 8 glasses together. Suppose I take five drops out of each (Circle each 5 with colored chalk.) and put them in a jar: How many are in the jar? 40. How many are left in each glass? 60. And how many glasses all together? 8. How many drops of water are left in the glasses? 480. So, now I have 1300 drops in jugs, 480 drops in glasses and 40 drops in the jar. How many drops all together? 1820. (The children figured all of this out without any paper and pencil, and I pointed that out to them. Some members of the class are quite capable of doing 28 x 65 using the conventional algorithm, paper, and pencil; for them, this was a lesson about thinking rather than about doing.)

After we finished this illustration of 28 x 65, one of the girls in the class came up with another way of thinking about it (and that's just what she called what she was about to tell me). I followed along and also put her explanation on the board so as to give it equal weight in the eyes of the class. She said, "I thought you could have three jugs. Two would have 650 drops in them, just like you said there. But if you put 650 drops in the third one, you'd have too much. You'd have to take out two glasses, because there are not 30 glasses in the story but only 28. And each glass has 65 drops, so you'd have to take out 65 plus 65 or 130 drops. Then in that third jug, you'd have 650 - 130 which is 520. You'd have to add that to the other two, 1300 + 520, and you get 1820."



I include her story to provide evidence that kids do think this way, and to support the argument that what they really need to be competent at school mathematics is some help connecting what they think with the conventional symbols and procedures. (Whether they should have to learn to do that in order to be good mathematicians is another question.)

One might argue that my drawings and the decompositions that underlie them are a structure imposed by an authority from outside on children's ways of thinking about mathematical processes. But there is an essential difference between the kind of structure I've attempted to put on kids' thinking about multiplication and teaching them to use the conventional algorithm. My structure is derived from the idea that numbers represent quantities and that mathematical symbols represent operations on those quantities. It makes a bridge between concrete representations and abstract symbols. In order to understand the difference between what I did with drawings and stories, and teaching children to use the conventional algorithm, think for a moment about what one does in order to get through a multiplication like $\begin{array}{r} 48 \\ \times\ 62 \end{array}$ using the conventional algorithm. (It will help the reader to make sense of what follows to do this multiplication using the method you were probably taught in school.)

Consider the first step in doing this computation (2 x 48), and think about all the assumptions involved in carrying that little one up on to the top of the four. Why do you multiply 2 x 4 and then add the one? Why not add the one to the four first and then multiply? (2 x 5) Kids ask these questions, if not "out loud" then in the errors they make as they try to use this process. But when we do this multiplication mechanically, we do not stop to remind ourselves that the 4 is really forty, and so when we multiply 2 x 4 to get 8 and add the 1 from the 16 we are actually adding 80 and 10 and we've got to put the answer down in some way that represents 90. We hold aside any thoughts about what that "little one" on top of the 4 might mean and just go on with the process, sometimes adding and sometimes multiplying, putting down the results where we were told to put them.

Now look at the second multiplication: 60 x 48--or did you think it was 6 x 48? In order to do it, we must "forget" that the 6 is really 60 and multiply it by 8 to get 48 and then know that we're supposed to put the 8 underneath the nine and put a little four up on top of the little 1 up on top of the 4 in 48. What can all that possibly mean? Now then, just in case that was not confusing enough, we multiply 6 (which is really 60) by 4 (which is really 40) to get 24 (which is really 2400) and we add that little 4 up there (which is really 400! How did it get to be 400? It is in the tens column after all???) to get 28 (which is really 2800).*

It is clear that if we did stop along the way to remind ourselves of what the numbers really mean, as I have done here, we would get horribly confused. The message is, therefore: When you are learning to multiply

---

*Tom Lehrer could probably sing it. He's also a math teacher.

big numbers, it's better not to think about what these numbers mean if you want to get it right! But if you don't know what the numbers mean and you forget what to do next, there is no reference for answering the question "what makes sense to do next?" By contrast, if you are using the decomposition and grouping steps I've described, you can stop anywhere along the way and figure out what the numbers mean as a way of figuring out what to do next. You do not have to attach those numbers to objects in order to make sense of them; the numbers themselves can be written in symbolically meaningful ways, so that if a 4 is supposed to mean 400, it looks like 400. By the time kids are in fourth grade, they don't need to know if it's 400 apples or 400 oranges, but they do need some help with what the symbols mean. The presence or absence of a zero or two is no trivial matter.

If the ultimate goal of math education is to get kids to be able to do the conventional algorithm and know what it means, the step by step grouping process which I have outlined can be seen as a "transitional algorithm," and this is how such processes are often presented in textbooks. If kids had a solid grounding in what multiplying numbers like 73 x 86 means, and they had practice in doing these multiplications in meaningful ways, it would probably not be a big transition to understanding the steps in the conventinal procedure. But fourth grade math is often taught by adults, who like most of us, can use the conventional algorithm but do not know what it means. So when teachers are making the decision about what lessons in the book to leave out because there is not time in the year to teach everything, the so called "transitional algorithms" probably get short shrift.

Making the connection between what is really "happening" to those numbers when you multiply them and the way the conventional algorithm represents what is happening requires relatively sophisticated knowledge of mathematics and the confidence that goes along with it. If you are not very careful in making that connection--using language, concrete representations, and symbols very clearly and precisely--children probably will get more confused than they would have if you just taught them the rules of the conventional process and had them practice it over and over and over. The danger for confusion may be even greater if one of their teachers teaches for meaning and another teaches for mechanical competence. So the more knowledgeable teacher (and anyone else who seeks to improve mathematics education) faces a dilemma: should we raise questions about or even ignore the conventional process at the risk of causing confusion? Or should we strengthen the child's capacity to use the conventional process by endless rote practice, even if he or she does not understand why it works? Choosing the later course locks us into a view of mathematical knowledge which requires the learner to "take it on faith." It presents mathematics, not as a subject that "makes sense" -- not as a way of thinking that you can figure out for yourself if you really work at it -- but as a set of rules for manipulating numbers that someone else made up and it would be better not to think about too much. What is at stake is the learner's attitude toward what is to be learned, and it is not just a matter of getting someone to "like" mathematics. It is a more serious matter of whether children have the opportunity to explore what mathematics really is and acquire competency in mathematical thinking.

Coin Problems

"Using only two kinds of coins, make $1.00 with 19 coins."

Early in the school year, I gave my fourth graders the challenge of solving this problem with two goals in mind, neither of which was directly related to learning about multiplication. I wanted to demonstrate that there were many possible routes to the solution of a math problem and I also wanted to show them that math problems could have more than one right answer. At first, the students did not know where to begin. I was challenged to help them find a "way in" to this problem that would leave most of the thinking in producing a solution up to them. The activities which I designed turned out to involve several weeks worth of practice with various aspects of multiplication. The simplified "coin problems" I gave them to do required the same process of taking numbers apart and putting them back together again that is the essence of the distributive law (described above on page 11). I did not tell them much about the connections between what we were doing and the more conventional multiplication procedure, although such connections could certainly be made. Instead, I capitalized on the fact that the coin problems did not look much like the arithmetic they had been learning from other teachers. These problems therefore had the potential to free them up to experiment and think mathematically rather than relying on mechanical routes to right answers.

As I thought about how to enable my students to "dive into" problems like the coin problem above, I began to wonder about the meanings of the words in the problem. Do all of these kids know what a coin is? And do they all have the more culture-specific knowledge that includes familiarity with the kinds of coins used in the United States? The relationship between a single coin and the amount of money it represents is a subtle one. Thinking about this relationship led me to examine the mathematical semantics that were the key to understanding how to solve these "coin problems." One dime is worth ten cents. One dime is in some sense "the same as" ten pennies, but a dime is only one coin, while ten pennies is ten coins. A dime could also be worth the same as six coins if one were a nickel and the others were pennies. So the total number of coins and the total amount of money are related, but they are related in complex ways. One coin might be worth one cent, but it also might be worth five, ten, twenty-five, or fifty cents. I know I needed to give my students some help in thinking about all this, and I realized they would learn some important things about how to work with numbers in the process.

I began my next lesson with a review of all the coins we use in the U.S., making a chart on the board which listed each coin's name and how much it was worth in cents. I also identified each coin with the first letter of its name so that I would not have to write the name on the board each time I wanted to refer to it. (Fortunately, all the coins' names begin with a different letter!) Although I knew it would not be of much use in solving the sorts of problems we would be working on, I included the silver dollar in the chart because the kids seemed to be intrigued by it.

I was careful to use the word "cent" when I was talking about the value of a particular coin and the word "penny" when I was talking about the coin that is worth one cent, even though these words are often used interchangeably.

COINS	AMOUNTS OF MONEY
penny	p = 1 cent
nickel	n = 5 cents
dime	d = 10 cents
quarter	q = 25 cents
half-dollar	h = 50 cents
silver dollar	s = 100 cents

This chart was particularly valuable to the children who were weak in English and less familiar with U.S. culture, but I made it a learning focus for all of the children by emphasizing the use of letters to stand for the names of coins. The more advanced students liked it because they thought it had to do with algebra. The next several times we did coin activities I put this chart up on the board, sometimes with the coin names, and sometimes using only the letters.

We began working with simple multiples: How much money is 3 dimes worth? 5 dimes? 7 nickels? 2 quarters? And then we did combinations: What are 3 dimes and 2 quarters worth? 7 dimes and 3 pennies? 2 quarters and 1 dime? (These were sometimes written on the board as 3d + 2q, 7d + 3p, 2q + 1d, and the use of these letters did not seem at all problematic to the children.)

Finding the monetary value of these combinations is an important exercise for setting a context in which the "order of operations" matters;

everyone knows that you multiply first and then add to get the value of the coins because of their familiarity with how money works. (I.e., to find the value of 3d + 2q, fist you multiply 3 x 10, then you multiply 2 x 25, then you add the results of these two multiplications.) This familiarity gave the students the opportunity to play around confidently in an area where they would later be introduced to more abstract forms. Because they knew how coins worked, they would be unlikely to add first and then multiply or make other procedural errors. The symbols that we use to indicate the "order of operations" in a mathematical expression are parentheses. Taught in the abstract, they constitute just another rule to be remembered: when you see (3 x 10) + (2 x 25) you remember to do the multiplications first and then add. But why? Perhaps by working with coin combinations, children can get practice with something that "makes sense" to them onto which they can later hang this rule.

Another sort of coin activity we did was figuring out how many of which kinds of coins would add up to a certain total. At first we used only pennies or only nickels or only dimes or only quarters or only half dollars, and we usually began by figuring out how many would make a dollar. These problems provided a benchmark for going on to more complicated problems. It is both practical and mathematically useful for students to be familiar with equivalents like: 4 quarters make a dollar, 5 nickels make a quarter, 20 nickels make a dollar, 10 dimes make a dollar, and so on. The multiplications and divisions here all involve

fairly "easy" combinations of 5 and 10, but there are a many variations.* Children can play around more easily with the processes of decomposition and regrouping because they do not have to strain to remember the "number facts" involved in the computations. The "problems" are ones they <u>can</u> solve and yet there is enough of a challenge to it to make it satisfying when they find the solutions.

Once you begin to use <u>two</u> kinds of coins to add up to a certain amount of money (e.g. nickels and/or pennies to make 82 cents) the problems become more interesting and fun because there are multiple solutions. The children seemed most enthusiastic about figuring out how many different ways one could make an amount like "82 cents" using nickels and pennies, and their enthusiasms gave me many opportunities to help them generate their solutions using strategic rather than random thinking. The nature of these problems is such that once you come up with one solution (like 16 nickels and 2 pennies for making 82 cents) you can make more solutions by trading: everytime you take five away from the pennies pile and you need to add one to the nickels pile, and vice versa. This operation gets at the basic concept behind place value; it also teaches kids that looking

---

*When children are figuring out how many quarters are in a dollar, what they are doing is the arithmetic process we call "division": figuring out how many 25's are in 100. It might be useful to make the connection between the work they need to do to figure that out and the conventional symbols for representing the operation, but I didn't choose to do that at this point. A connection would be especially fruitful, however, in getting some meaning attached to the idea of a "remainder." When trying to figure out, for example, how many nickels are in 37 cents, it is not hard to see that 35 cents of that can be turned into nickels, but that there are 2 cents that cannot. If you only turn 25 cents of it into nickels, you will have 12 cents as a remainder, and it's clear you can get more nickels. Using the coin trading processes thus provides a representation of a mathematical idea in a media with which children are familiar.

for patterns and relationships is a useful strategy in solving a problem they care about. The process gets children involved in mathematics in a way that enables them to challenge themselves to find solutions, constructing mathematical hypotheses along the way about whether they have found them all yet. Students who are not yet able to use multiplication and division can find solutions by adding, and so a lack of computational "skills" does not hinder their participation. On the contrary, they develop their own more sophisticated adding strategies which come to look more and more like multiplication.

As my class worked on these problems, I encouraged them to "guess and check:" that is, to pick numbers of coins that they thought might add up to the desired total, and then to check to see if they actually would. Once they got going on this procedure, they were able to think mathematically and modify their guesses strategically. But first I had to find a way to help them overcome a habit that inhibited their use of this way of getting a solution: once children in school recognize that something they have put down on paper is wrong, they are intent on forgetting it and erasing it. They don't want to leave any evidence around of their "failures," especially once thay have figured out how to "succeed." This way of approaching mathematics is not only conceptually problematic; it causes logistical difficulties as well.

I needed to find a way to reward students for all of their guesses so that they would record their "wrong" guesses and think about why they did not work, so I gave them a framework for writing down their thinking about the coin problems. For example, for a problem like "How many ways can you

make a dollar using dimes and/or nickels?" I directed them to put the following down on paper:

d	n	total amount of money

I suggested that they list <u>all</u> of their attempts, not just the ones that "come out right," and I directed them to put a line through the ones that did not satisfy all the conditions of the problem. I praised them for their numbers of tries, not just for the number of solutions they could find that met all the conditions.

After several minutes of working, one student's paper looked like this:

d	n	total amount of money
5	10	$1.00
~~6~~	~~9~~	~~$1.05~~
6	8	$1.00
~~7~~	~~7~~	~~$1.05~~
7	6	$1.00
8	4	$1.00
9	2	$1.00

I would assume, from looking at such a paper that the student first thought he should decrease the number of nickels by one if he increased the dimes by one. It took him two trials to figure out that the nickels had to be decreased by <u>two</u> if the total has to come out to $1.00.

I wanted to get the message across that it was okay to try something you thought might work even if you were not sure it would. This is the mathematician's work of "hypothesis testing" on a very simple level. It gets kids going on problem solving with less fear of "being wrong," and thus enables them to be more confident in their own abilities to eventually figure something out. It is important to have them practice this process on something they actually can figure out so that it is not just an exercise in teacherly reassurance.

I asked the students to talk about the "patterns" they noticed in their charts. On a problem such as the one illustrated above, they would say things like, "If the dimes go up by one, the nickels have to go down by two." or "Everytime you put in another dime, you have to take away two nickels." I tried to use their language myself in noting patterns and to make telling about patterns a central part of our work on problem solving. Each time we began a new problem, however, the discovery that there would be a pattern and the description of patterns was left up to the students. If they were working away randomly, I did not interfere by giving them a suggestion for an "easy way" to get a lot of solutions. If there was a strategic pattern obvious in their work, I took note of it and encouraged them to verbalize it.

An example of another kind of problem we did which focused on grouping and the recognition of strategic patterns was finding all the ways you could make $1.00 using only two kinds of coins, for example, nickels and quarters, and then to find all the ways to make $2.00 using the same two kinds of coins. In the first part of this problem, many students observed

that the numbers of nickels were always multiples of five. Many figured out after a while of experimenting, that you could take the solution to the first problem and make solutions to the second problem either by adding 20 to all the nickel amounts or by adding 4 to all the quarter amounts.

After a few days of working with two kinds of coins to get a given total amount of money I introduced problems using 3, 4, 5, and 6 different kinds of coins. These problems offered many more possibilities for "guesses" and it was harder to see the patterns. It also required thinking about relationships among three variables: the number of coins; the number of kinds of coins; and the value of the total.

Some children seemed ready to talk about the question, "How do you know if you've got all the solutions?" and they explained their theories about this to me with varying degrees of clarity. What was most important was not the content of their explanations but that they believed there was a way to tell if you had them all and that they could figure it out.

In all of the problems I've described so far, there was no condition placed on the total number of coins that could be used. To make a dollar, one could use 20 coins or 2, as long as their values added up to 100 cents. We also did some problems in which the number of coins that could be used was specified, but the kind of coin was not: e.g. "Make $1.00 using 3 coins." (One could also make up problems where the question was: How many different amounts of money could you make using a given number and given kinds of coins?)

Considering each of these kinds of problems separately and practicing each one on several different examples helped the students to develop the confidence, the skill, and the organization to approach a more complex coin problem like the one with which I began this section: "Using only two kinds of coins, make $1.00 with 19 coins."This problem sets limits on the total number of coins to be used, the amount of money they must add up to, and the number of different kinds of coins that can be used. The sorts of problems we had been working on so far had set limits on only one of these three areas. Students' solutions to more complex problems were organized using the same chart described above, with extra columns for the total number of coins and the number of kinds of coins when these conditions were part of the problem.

One problem we worked on was: "Use a total of 24 coins to make $1.00." One of the children's charts looked like this at the beginning stages in her work:

	p	n	d	q	h	total amount of money	total number of coins
(Solution #1)		5	5	1		$1.00	11
(Solution #2)	25	5	5			$1.00	35
(Solution #3)	20	6	5			$1.00	31
(Solution #4)	15	7	5			$1.00	27
(Solution #5)	10	8	5			$1.00	23

This student is getting closer and closer to a solution that meets both of the conditions. Solution #1 has too few coins and solution #2, too many even though they both total $1.00. Her thinking then seems to focus on the pennies rather than quarters in solutions #3, #4, and #5, perhaps because 35 seems closer to 24 than 11 did, perhaps because pennies

give you more alternatives with less impact on the total than quarters do. Solution #4 still has too many coins, and #5 has too few, but they are approaching 24 very closely. At this point, the student might observe that with each of the changes from pennies to nickels, the total number of coins goes down by four. Recognizing this means it's not likely that you will find a solution if you keep messing around only with pennies and nickels. She might then see that it is a short step to the solution if you exchange a dime for two nickels, but she might also start off on an entirely different track.

There is a great deal of mental "trading" going on in these problems that is similar to what we ask students to do in a more abstract way when they work with place value, e.g. carrying and borrowing. There were remarkably few errors in the students' work at the trading level; when they took something away from the dimes column, they put the appropriate something else down in the nickels column. The amount of self-monitored strategic thinking varied among the students as they "guessed" and "checked," but they were all able to proceed through several trials until they found at least one solution. They continued to work on these kinds of problems after class and often came in the next day with more solutions; they also challenged their friends and parents to find even more. Some of the work the students did is illustrated in figures 7 and 8. Figure 8 is interesting because it shows that the student used rather primitive adding strategies to find solutions, but this did not seem to affect her interest or success.

Figure 7

p.	n.	d.	q.	h.d.	total of coins	amount money	right or wrong
10	10	4			24	$1.00	right
20	3			1	24	$1.00	
20	1		3		24	$1.00	
20	=	3	=	1	24	$1.00	right
10	10	4			24	$1.00	right
20	5	2	2		24	.90	

BEST COPY AVAILABLE

P	N	D	Q	amount	amount coins
10	10	4		1.00	24
20		3		$1.00	24
15	1	5		$1.00	24
20	1		3	$1.00	24
8	12	3	2 done	$1.00	24
10	9	6		$1.18	24
5	9	5		$1.05	24
5	9	5		$1.05	24
10	9	5		$1.05	24
1	8	5		$1.06	24
10	8	5		$1.05	24
21	1		1	$1.01	24
10	16	5		$1.01	24

figure 8

BEST COPY AVAILABLE

To make the connection between these "coin problems" and the grouping and counting processes involved in multiplication, I directed the students to apply their thinking about combinations of coins to problems like: "How much money would you need to buy 8 candy bars if the candy bars cost 13 cents each?" With the goal of having them think about this problem rather than solving it, I set a sub-problem: "What are some combinations of coins that would add up to 13 cents?" I wrote their answers on the board using letters for the coin names:

3p + 1d for one candy bar
3p + 2n for one candy bar
13p for one candy bar
1n + 8p for one candy bar

Taking each of these separately, I asked them how many of such coins would be needed to buy 8 candy bars; i.e. if you could get one candy bar for a dime and 3 pennies, how many dimes would you need for 8 candy bars? and how many pennies? I wrote 8d + 24p on the board as they answered, and then asked them to figure out how much money that was. 80 + 24 = $1.04, so 8 candy bars would cost $1.04. Then we went through the same process with the next combination of coins: nickels and pennies. You'd need 2 nickels and 3 pennies for one candy bar. How many nickels for 8 candy bars? and how many pennies? The total would be 16n + 24p or 80 + 24 or $1.04. Suppose you used all pennies? 13p for one, 104p for 8, or a total of $1.04. We would discuss whether it was easier to figure out the total

using dimes and pennies or nickels and pennies or all pennies, and different children had different good reasons for thinking one or another way of thinking about it was "easier."

My major interventions in their work were to emphasize trial and error as a legitimate problem solving method and to give some structure to it, to recognize multiple acceptable solutions to the same problem, and to have conversations with the children about the strategic patterns in their work. To help them focus on strategies and patterns, I also gave them assignments like those illustrated in figure 9 and 10. They completed these sheets independently and, when they were all finished, we had a class discussion of "what they noticed" as they were working. I also had some individual conversations with children as they worked on these to point out potentially useful strategies, especially if I observed that they were not using the relationships among the amounts to help them fill in the blanks. Many children seemed unfamiliar with the idea that one might use an answer to one question to help figure out the answer to another in this classroom "worksheet" context. (For example, they did not necessarily see that they had just figured out the value of 8 quarters and so in order to figure it out for 9 quarters, all you had to do was add 25 instead of starting from scratch.)

It is worth noting that we never used actual piles of coins or "play money" to work on these problems, although there were at least two children in the class who might have been more involved if we had been working at that concrete level. In the solutions to the problem illustrated on page 43, the step that would lead to a solution that met

Figure 9

Name Rachel Parton
Date 10-1-84

How much money would you have if you had:

4 quarters $1.00	5 quarters $1.25
6 quarters $1.50	7 quarters $1.75
8 quarters $2.00	9 quarters $2.25
10 quarters $2.50	11 quarters $2.75
12 quarters $3.00	13 quarters $3.25

Do you see a pattern? Yes
Can you figure out how many quarters it would take to make 7 dollars 28
12 dollars 48

How much money would you have if you had:

4 nickels 20¢	10 nickels 50¢
5 nickels 25¢	7 nickels 35¢
9 nickels 45¢	17 nickels 85¢

8 nickels 40¢	7 nickels 35¢
16 nickels 80¢	14 nickels 70¢
32 nickels $1.60	28 nickels $1.40

Do you see a pattern? NO
Can you figure out how many nickels it would take to make 7 dollars 140
12 dollars 240

BEST COPY AVAILABLE

Figure 10

Name Sadie Miles

Date 10-2-84

How much money would you have if you had:

20 nickels $1.00
25 nickels $1.25
30 nickels $1.50
35 nickels $1.75
40 nickels $2.00

21 nickels $1.05
26 nickels $1.30
31 nickels $1.55
36 nickels $1.80
41 nickels $2.05

Do you see a pattern? Yes

Can you figure out how many nickels there would be in 3 dollars 60
6 dollars 120
10 dollars 200

How much money would you have if you had:

3 quarters $.75
6 quarters $1.50
9 quarters $2.25
18 quarters $4.75

7 quarters $1.75
14 quarters $3.50
28 quarters $6.50
30 quarters $7.00

Can you figure out how many quarters there would be in 3 dollars 12
6 dollars 24
10 dollars 40

BEST COPY AVAILABLE

both conditions might have been very obvious if there were piles of coins at hand. However, more conceptualization of the implications of trading was probably required because they did not have the concrete materials, and most of the children seemed ready for that level of intellectual work. Not having actual coins with which to experiment required them to find ways to represent to themselves "what would happen if. . ." as a basis for deciding which coins to try next.

Multiplication in Real-Life Problems

The grouping and exchanging that occurred in the coin problems is based on children's real-life experiences with coins. These problems provided an opportunity to draw on experiences out of school as a basis for developing mathematical knowledge in the classroom. Having acquired that knowledge, children can go back to real-life situations and deal with money even more proficiently. This is one way to connect classroom learning with so-called "real-life problems." Purists would say, however, that a mathematical problem is not a real-life problem for a child unless it arises from circumstances in which the solution matters to him or her. Such a problem would arise, for example, when a child needs to figure out if there is enough money in the piggy bank to purchase a coveted toy. In this siutation, the child needs to do math to get what he or she wants. But there are many other senses in which the solution to a problem could be said to "matter" to a child. For the purposes of this paper, a real-life problem in mathematics will be defined as one which involves circumstances that are interesting and familiar to children, is connected to elements of the life that go on outside of the schoolroom, and requires mathematics (not just computation) to solve it.

I will describe two such problems which intrigued my fourth graders: figuring out how many children of their size would weigh one ton and figuring out the total weight of the "live load" on our classroom floor (i.e. of everything in the room that was not a part of the building itself). The first of these is like the coin problem; it draws on

children's real experience, and working on it will make them more skillful in solving real problems. The second arises more directly from children's own concerns about how a problem can be solved in the world outside the classroom.

Our work on the "ton problem" began with something the children had heard on the news about crops grown in Michigan. It was clear to me from what they were saying that they did not have much sense of how much a ton actually weighed. Seeing an opportunity for a math lesson that would connect with the real world, but being unable to procure a ton of either corn or soybeans, I suggested we figure out how many children their size would be needed to make a ton. The problem to be solved was thus set by me, but in terms that engaged the students' concern to find a solution.

The "live load problem" was part of a larger exploration by the class of the engineering problems that arise in the building of bridges. This project began with some wondering about a bridge under construction in Zilwaukee, Michigan that persists in falling down despite engineers' attempts to get it standing. The problem to be solved in Zilwaukee became a problem of concern to my fourth grade students because it made them worry about whether the bridges <u>they</u> drove over were also going to fall down. Many of them are familiar with the Mackinac bridge, a most impressive structure that connects the Upper and Lower Peninsulas of Michigan; this bridge is very long and very high and the water underneath it is very cold and very turbulent. Of course, we were not "solving" the actual problem of keeping the Mackinac bridge from falling down, but our work has definitely contributed to the children's understanding of how mathematics can be useful in figuring out how to make it strong. Our solution to the "ton" problem could also be said to contribute to

children's understanding of real-life mathematics rather than producing an outcome that was more directly useful. But both problems (and several others of their kind) can serve to make a fruitful connection for children between what needs to be learned in classroom and knowledge that is useful to have outside of classrooms.

"If 631 tons of soybeans and 325 tons of corn had been produced in 1984 as feed for animals raised in Michigan, about how many tons of feed was that all together?" Estimation was an idea my students were familiar with and most of them confidently answered that "it was about 900." I wrote an addition problem on the board so we could figure out the total more precisely:

```
  325  tons of corn
+ 631  tons of soybeans
  956  tons of animal feed
```

One of the more eager and technically competent boys in the class then volunteered the information that a difference of 56 tons (between 900 and 956) was a difference of 112,000 pounds because there are 2,000 pounds in a ton. I knew that most of the class probably did not understand what he was talking about, and for a complex set of social reasons, I wanted to seriously acknowledge his contribution. I also decided that unless everyone in the class had a clearer idea of what a "ton" might be, it would not be appropriate to go on to discuss whether 956 tons was a very exact measure.

I wrote on the board: "2,000 pounds = 1 ton," to give the students a way to begin to develop a concrete sense of what that meant. I then posed the following problem: "One person in this class probably weighs about 50

pounds. How many of you would it take to make a ton?" They looked at me blankly, and with the teachers' ever-present awareness of the shortness of time and multiplicity of goals, I came up with a way to get them going on the problem. I wrote on the board again, using the equals sign to indicate a weight relationship:

1 person = 50 pounds 2 persons = ?

Several students called out "100" Then I wrote on the board : 10 persons = ? and by this point, they were almost all drawn in to working on the problem, although they probably did not yet see how it was going to get us to figuring out how many persons their size would be in 1 ton. Several students figured out that 10 persons would weigh 500 pounds, and I asked them to explain their methods. Some added 50 ten times; some added 100 five times, explaining that 10 was 5 x 2, some just "put a zero on the end of 50 because that was what happened to any number that got multiplied by ten.

I added this information to the structure on the blackboard and that gave several children ideas about how to figure out how many 50-pound people would weigh a ton:

2,000 pounds = 1 ton

1 person = 50 pounds
2 persons = 100 pounds
10 persons = 500 pounds

One girl explained: "If you double 10 persons you get 1,000 pounds and that's 20 persons. If you double that, it's 40 persons and it makes one ton." Another explanation came from one of the boys: "If you double 2 persons you get 4 people and that equals 200 pounds. And so for 2,000 pounds, you just add a zero to both and its 40 people." The boy who had started it all in the first place by changing 56 tons into 112,000 pounds

then said, in his typically mechanical fashion, "You just divide 50 into 2,000 and you get 40." I accepted all these explanations as good ways to figure out how many 50 pound people would weigh a ton. The children were getting good practice with the grouping and counting principles we had been working on in the multiplication stories, and they were solving a problem that had come to matter to them.

A girl who had been patiently waiting to say something then offered me a terrific way to extend the problem. She said that she did not think that anyone in the class weighed as little as 50 pounds! (I had originally chosen the number because I thought it was both close enough to reality and easy to work with). Others chimed in, calling out their own weights, between 50 and 100 pounds. I decided to ask them what <u>they</u> thought a <u>reasonable</u> "average" for the class would be. It did not seem appropriate to present them with procedures for finding the mathematical average at this point, since I knew that would be a complicated lesson and it would take us away from the ton problem, which <u>they</u> now <u>wanted</u> to figure out. They agreed on 75 pounds as an average because "some people were above it and some people were below it;" that seemed like a mathematically appropriate explanation of an "average" for their level. I then asked the class whether <u>73</u> pounds would do equally as well, just to press their thinking a bit further. In response, I got several versions of the statement that 75 was just as good as 73 and it would be an easier number to work with.

As on many other occasions, these children seemed to be displaying quite a sophisticated level of practical mathematical reasoning. If one takes the time to listen to students and build on their ideas, there is very little risk of losing mathematical integrity. Many of the sensible

suggestions about how to solve this (as well as other similar problems) came from children in the class who were thought by other teachers to be "not very good at" mathematics. They participated in the process as much as and well as the recognized "whizzes" in arithmetic. This participation gave everyone in the class a new message about what it means to be good at math and who qualifies for that recognition.

Revising our work on the ton problem to accomodate 75-pound children provided more opportunities for good mathematical problem solving, as well as more practice with the concepts underlying multiplication. I now wrote on the board:

2,000 pounds = 1 ton 1 person = 75 pounds

and asked the class "what else we could put up on the board." I intentionally did not go directly to the question: "How many 75 pound people would make 1 ton ?" so as to keep everyone "with me" in this process as much as possible. Using the multiplicative structure I had introduced earlier, we established that:

2 people = 150 pounds
10 people = 750 pounds

and everyone was very anxious to tell me how they figured these out. I then challenged them with 20 people = ? Many of the students figured out that it was 1,500 pounds some by doubling 750; others, by multiplying 150 by 10. I made a "lesson" out of the different ways of getting the answer; the various grouping strategies the children used to derive from the same concepts that underline the algorithms we invent to multiply large numbers.

Once we had established that 20 people = 1,500 pounds, one of the boys recalled our earlier "solution" to the ton problem (i.e. 40 fifty-pound

people = 1 ton) and he observed, "You can't use 40 people because that would be too much." After getting him to explain his reasoning, I asked the class, "What shall we do?" Karen, a particularly reticent girl, lit up and said, "25 people!" I did not make any judgment of her "answer", but instead turned the job of assessing what she said over to the whole class. I asked <u>them</u> how much 25 people would weigh. The same girl figured out that if you added the (known) weights of 20 people, 2 people, 2 more people, and 1 more person, you'd get the weight of 25 people: 1,500 pounds + 150 pounds + 150 pounds + 150 pounds + 75 pounds = 1,875 pounds. In the following discussion, I asked questions and did some arithmetic, but the students did the problem solving:

T: So would 25 people make a ton?

Karen: No, it's too little.

T: How many more people should we add?

Tom: I think it should be 30 people.

T: How much would 30 people weigh?

Tom: Well, you add 375 pounds, because that's what five more people would weigh. Karen figured that out.

T: 1,875 + 375 is 2,250. Is that a ton?

Jim: No, it's too much.

Katy: Let's try 26. Just add 75.

T: 1,875 + 75 = 1,950.

Mack: 27!

Jenny: No, that would be over!

Esther: You'll never get it!

Chorus: Yeah!

T: Is there a way we <u>can</u> get to one ton?

Steven: My brother only weighs 50 pounds; we could add him!

(I was amazed and pleased by this return to reality after we had been playing around so intensely with pure numerical realtionships).

T: Great! Is there any <u>other</u> way we could make one ton? (always seeking to encourage multiple solutions).

Annie: You could use two twenty-five pound people.

Dennis: You could use a forty-pound person and a 10 pound person.

(Note: This method of generating solutions seemed clearly related to our work on the coin problems, although I didn't mention it to the class).

Peter: A 45-pound person and a 5-pound person?

T: Where would you find a 5-pound person?
(wanting to keep us in the realm of reality)

Annie: You could use a sack of potatoes or a new born baby. (!!!)

T: If you don't count teachers, and Steven goes to get his brother, are there enough people in this room to make about one ton?

Dennis: No, there are only 24, and you'd need 26.

T: How many more people is that?

Arthur: It's two more.

Steven: No, it's <u>three</u> more because you've got to count my brother, too.

This discussion proceeded at a rapid and exciting pace. Needless to say, I was pleased at the children's level of engagement in the problem, their attention to one another's ways of thinking, and their use of multiple and appropriate mathematical strategies. They built with confidence on the multiplicative structure I had given them to figure out how many people would weigh how many pounds until they reached what had become <u>their</u> goal: a ton.

The focal points of our class's study of engineering problems are two suspension bridges: the Brooklyn bridge and the Mackinac bridge. Both of these structures are salient to the children, one because they have been hearing the rather dramatic story of its construction as a daring experiment with the suspension design, and the other because it is relatively nearby and was the longest suspension bridge in the world when it was opened in 1958. The stories of the building of theses bridges make history and geography, as well as mathematics and science, come alive for fourth graders. The characters involved in building these bridges were rather colorful, and the risks they took are understood and appreciated by 10-year-olds.

How many people can go across the bridge at one time? How many cars? How do they know the cables are strong enough to hold up the road? What if a lot of heavy dump trucks full of gravel wanted to cross the bridge at the same time? What if it snowed and snowed and snowed and all the snow turned to ice? Would the bridge fall down from the weight of the ice the way the tree branches do? These are the sorts of things my fourth graders wonder about. The designers of the Brooklyn Bridge and the Mackinac Bridge used multiplication to address these problems. The Roeblings built a structure across the East River that was six times stronger than had been necessary at the time: each of the 1,520 suspenders that attached the bridge floor to the cables was capable of sustaining a weight of more than 50 tons.* David B. Steinman, the designer of the Mackinac bridge, dealt with the potential for ice pressure on the bridge by considering the greatest possible

---

*Judith St. George, The Brooklyn Bridge: They Said It Couldn't Be Built (N.Y.: G.P. Putnam's Sons, 1982), p. 22.

pressure that could be built up (23,000 pounds per lineal foot) and simply multiplying this figure by five; he convinced people the bridge would be safe by engineering it to withstand the hypothetical, impossible ice pressure of 115,000 pounds per linear foot.**

How does one begin to understand (and help children to understand) the sort of problem solving and mathematical "figuring" that goes into estimating the potential "live load" on the Brooklyn Bridge or Mackinac Bridge? We could not solve that problem in the confines of a school lesson, but we could find out what might be involved in figuring out the total weight acting on the classroom floor.***

This is a complicated activity for fourth graders, and their engagement in it necessarily distracts them from the more direct discussion of bridge-related problems. While they were doing it, I supervised their activity, asked probing questions, and reminded them every once in a while of how what we were doing connected back to their questions about bridges.

The weighing they needed to do to find the live load on the classroom floor involved the class in other sorts of problem solving. At first, the children wanted to weigh all of the people in the room individually on the bathroom scales that I had brought in. But then one of them took note of the fact that people who were not always here came in and out, people with different weights were in the class in other years, and on parents' night, the room would be full of heavy grown-ups. Everyone

---

**Lawrence A. Rubin, Mighty Mac: The Official Picture History of the Mackinac Bridge (Detroit: Wayne State University Press, 1958), p.16.

***The idea and structure for this activity were adapted from Mario Salvadori and Michael Tempel, Architecture and Engineering: An Illustrated Teacher's Manual on Why Buildings Stand Up (N.Y.: new York Academy of Sciences, 1983), p.4.

settled on the job of finding the live load on the floor "right now" and they put off speculating about an appropriate "maximum" until later. (This speculation involves some of the same kind of "multiplication" that David Steinman did to cope with the ice pressure on the Mackinac Bridge). After this discussion, their strategy for finding the contribution that people's weights would make to the live load was different. They no longer wanted to weigh everyone in the room individually and add the weights together; instead they settled on taking a "sample survey" (i.e. asking the nearest four kids what they weighed) and getting a rough average, then multiplying that average by the number of kids in the class. (The three adults in the room were asked their specific weights, however, and these were added to the total).

When the students turned their attention to the furniture, there was no talk of weighing each desk and chair separately. The desks and chairs were sorted by size and the number of each size counted. Then one member of each size group was weighed and the appropriate multiplication carried out. Some of these multiplications were done using the conventional algorithm, others, using various other grouping and adding strategies.

They tried to put the chairs on the scales, but "they wouldn't stay" and there was some argument about whether it made the chair "lighter" if two people tried to balance it. One child came up with the idea that she could stand on the scale and hold the chair and subtract her weight from the total (a 1-2-1 contact fan, no doubt). A great discussion then ensued about how to weigh the desks since one desk was too heavy for one child to hold. I offered the possibility of using two scales without giving any suggestions about how to use them. They figured that the desk could be held up by two students who could each stand on one of the two scales, but

they disagreed a lot about whether each of the holders was carrying an equal weight and about how to find the weight of the desk. After a while, after doing some weighing and more talking, they agreed that it didn't matter if they were holding equal weights, or if it did, they were going to ignore it. They settled on the idea that two students could each stand on one of the scales, holding a desk between them. They added the weights of the two students before they picked up the desk, then subtracted that from the total shown on the two scales after they picked up the desk and that was the weight of the desk. I think they settled on this strategy at least in part because it turned up a number which seemed to them to be about what a desk should weigh.

Here is where narrative catches up with reality. Tomorrow the class will try to figure out how to estimate the weights of the considerable numbers of books, papers, and pencils in the room, and then the next job will be to find a strategy for approximating the weights of the larger pieces of furniture which cannot be moved.

Conclusions and Recommendations

Fourth graders can and do think mathematically. They have the capacity to gather information, to organize it strategically, to generate and test hypotheses, and to produce and evaluate solutions. They can talk about what they are thinking, they can listen to and appreciate another student's way of understanding something, and they can invent problem-solving procedures that are both useful and sensible.

What sort of help do children need from adults in order to be able to do these things and to be confident in their ability to do them? They need to be asked questions whose answers can be "figured out" not by relying on memorized rules for moving numbers around but by thinking about what numbers and symbols mean. They need organizational structures on which to hang their thinking so that they can see it in a form that leads to new discoveries. They need to be treated like sense-makers rather than like rememberers and forgetters. They need to see connections between what they are supposed to be learning in school and things they care about understanding outside of school, and these connections need to be related to the substance of what they are supposed to be learning.

Very few of these needs are being met in mathematics classes in elementary schools. Like many other subjects, math is treated as a body of information that teachers give to students about how to do something. Students either learn it or they don't; they get the answers right if they've learned it and wrong if they have not. There are certainly some skills that can be learned in this way, but mathematics canot be learned unless children see how it is done, and doing mathematics is more than applying rules.

There are two big stumbling blocks to bringing about the kind of change that would be indicated in the math lessons I have described in this paper. One is the current culture of schooling in which procedures for producing right answers are more likely to be valued than the process of making sense of those procedures. The second is the minimal knowledge of mathematics possessed by most elementary school teachers. It is difficult for them to judge what is entailed in doing a good job at teaching mathematics because they have so little experience doing mathematics. These two problems are related in complex ways, and their solutions will be a challenge to anyone who seeks to improve the situation.